John Connett's Master's Exam[1]

The questions:

1. There are an infinite number of primes.
2. Prove that the geometric series converges and find its sum.
3. Derive the quadratic formula.
4. Prove the Mean Value Theorem.
5. Prove the Law of Cosines.
6. Prove that $\sqrt{2}$ is irrational.
7. There is no surjection from a set to its power set.
8. Prove Lagrange's Theorem.
9. Prove that any uncountable subset of the reals has a limit point.
10. Prove the harmonic series diverges.

My solutions:

1. There are an infinite number of primes.

   PROOF. Suppose there are only finitely many of them, say $p_1, p_2, \ldots, p_n$. Then consider $P = (p_1 \cdot p_2 \cdot \ldots \cdot p_n) + 1$. The remainder of the quotient $\frac{P}{p_i}$ is 1 for each prime p_i, so P is not divisible by any prime. Then it cannot be divisible by any composite number, so P is prime. But $P > p_i$ for each prime p_i, so it cannot be prime. This is a contradiction. Therefore there are an infinite number of primes. $\square$

2. Prove that the geometric series converges and find its sum.

   PROOF. The geometric series is the sum $\sum_{i=0}^{\infty} x^i$ for $|x| < 1$. Consider the limit of the k^{th} partial sum:

$$\begin{aligned}
\sum_{i=0}^{\infty} x^i &= \lim_{k\to\infty} \sum_{i=0}^{k} x^i \\
&= \lim_{k\to\infty} \left[1 + x + \ldots + x^k\right] \\
&= \lim_{k\to\infty} \left[\frac{1 - x^{k+1}}{1 - x}\right] \\
&= \frac{1}{1 - x} \qquad \text{since } |x| < 1
\end{aligned}$$

$\square$

[1] Robert Wheeler shared this with me.

3. Derive the quadratic formula.

PROOF. Suppose $ax^2 + bx + c = 0$ where $a \neq 0$. Then we complete the square to see:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\ x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} &= \frac{b^2}{4a^2} - \frac{c}{a} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\ x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

$\square$

4. Prove the Mean Value Theorem.
To prove this, first we establish two other results.

THEOREM 1. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f reaches its maximum and minimum on $[a, b]$.*

Proof: The continuous image of a compact set is compact[2], so $f[a, b] \subseteq \mathbb{R}$ is compact. By Heine-Borel, this means it is closed and bounded. Let $\hat{y} = \sup f[a, b]$. Boundedness means $\hat{y} < \infty$, and closure means $\hat{y} \in f[a, b]$. Then there exists $\hat{x} \in [a, b]$ such that $f(\hat{x}) \geq f(x)$ for all $x \in [a, b]$, i.e. f reaches its maximum at $\hat{x}$. Similarly, f reaches its minimum. $\square$

THEOREM 2 (Rolle's Theorem). *If $g : [a, b] \to \mathbb{R}$ is continuous, its derivative is continuous on (a, b), and $g(a) = g(b) = 0$ then there exists $c \in (a, b)$ such that $g'(c) = 0$.*

Proof: If $g \equiv 0$ on (a, b), choose c to be any point in (a, b). If g is not identically zero, assume without loss of generality that at some point $x_+ \in (a, b)$ we have $g(x_+) > 0$. Using the prior result, let $c \in [a, b]$ such that $g(c) \geq g(x)$ for all $x \in [a, b]$. Note that $c \in (a, b)$ since $g(c) \geq g(x_+) > 0 = g(a) = g(b)$.
Now we wish to show $g'(c) = 0$. We know $g'(c) = \lim_{x \to c} \frac{g(c) - g(x)}{c - x}$. Consider the left and right hand limits. As $x \to c$ from the left, $g(c) \geq g(x)$ so $g(c) - g(x) \geq 0$ and $c \geq x$ so $c - x \geq 0$, thus $g'(c) \geq 0$. Similarly, as $x \to c$ from the right, $g(c) \geq g(x)$ so $g(c) - g(x) \geq 0$ but $c \leq x$ so $c - x \leq 0$,

[2]This can be shown using the covering definition of compactness.

thus $g'(c) \leq 0$. We know $g'(c)$ exists, so these left and right hand limits are equal, therefore $g'(c) = 0$. $\square$

THEOREM 3 (Mean Value Theorem). *If $f : [a, b] \to \mathbb{R}$ is continuous and its derivative is continuous on (a, b) then there exists $c \in (a, b)$ such that*

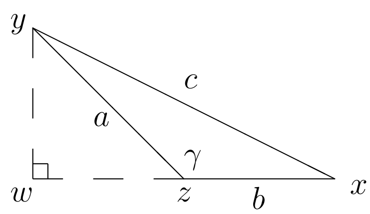
$$f(b) - f(a) = f'(c)(b - a)$$

PROOF. Let $g(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}(x - a) + f(a)\right]$. Note that $g(a) = g(b) = 0$, $g \in C[a, b]$ and $g \in C^1(a, b)$. So by Rolle's Theorem, there is a $c \in (a, b)$ such that $g'(c) = 0$. Then $0 = f'(c) - \frac{f(b)-f(a)}{b-a}$, thus

$$f(b) - f(a) = f'(c)(b - a)$$

. $\square$

5. Prove the Law of Cosines.



PROOF. If we denote the length of segment $\overline{wy}$ by wy, we see that

$$\begin{aligned} wy &= a\sin(\pi - \gamma) = a\sin\gamma \\ wz &= a\cos(\pi - \gamma) = a(-\cos\gamma) = -a\cos\gamma \\ wx &= b + (-a\cos\gamma) = b - a\cos\gamma \end{aligned}$$

Then by applying the Pythagorean theorem,

$$\begin{aligned} c^2 &= (wy)^2 + (wx)^2 \\ &= a^2\sin^2\gamma + (b^2 - 2ab\cos\gamma + a^2\cos\gamma) \\ &= a^2(\sin^2\gamma + \cos^2\gamma) + b^2 - 2ab\cos\gamma \\ c^2 &= a^2 + b^2 - 2ab\cos\gamma \end{aligned}$$

$\square$

6. Prove that $\sqrt{2}$ is irrational.

LEMMA 1. *a^2 is even $\Rightarrow$ a is even.*

Proof: Suppose a is odd. Then $a = 2k - 1$ for some $k \in \mathbb{N}$. Then $a^2 = (2k - 1)^2 = 4k^2 - 4k + 1 = 2(2k^2 - 2k) + 1$, so a^2 is odd. This contradicts our hypothesis, therefore a must be even. $\square$

PROOF. Suppose $\sqrt{2}$ is rational. Let $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{N}$ and $\frac{p}{q}$ is in lowest terms. Then $2 = \frac{p^2}{q^2}$ and $2q^2 = p^2$ so using the lemma we see p is even. Let $p = 2k$. Then $2q^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$, thus q is even. This contradicts our assumption that $\frac{p}{q}$ was in lowest terms, so $\sqrt{2}$ is not rational. □

7. There is no surjection from a set to its power set.

PROOF. Consider a set X. Suppose there is a surjection $f : X \to \mathcal{P}(X)$. Let $E = \{x \in X | x \notin f(x)\}$, i.e. the subset of X of consisting of elements not mapped to sets containing themselves. Since E is a subset of X, it is in the image of f, so there is an x_0 such that $f(x_0) = E$.
Either x_0 is in E or not. If it is, then by the definition of E, $x_0 \notin f(x_0) = E$, which is a contradiction. If it is not in E, then $x_0 \in f(x_0) = E$, which is a contradiction. Therefore there cannot be such an x_0 that is mapped to E. Thus f is not onto, therefore there cannot be such an f. □

8. Prove Lagrange's Theorem.

THEOREM 4. *If $A \leq G$ a finite group, then $|A|$ divides $|G|$.*

PROOF. Consider cosets gA of A.

Claim: Each coset has the same number of elements. We wish to show that for each $g \in G$, $|gA| = |A|$ by showing that multiplication by g is a bijection. Clearly it is onto its image. Suppose $ga_1 = ga_2$. Then $g^{-1}ga_1 = g^{-1}ga_2$ [inverses in a group are unique] thus $a_1 = a_2$, so multiplication by g is one-to-one. Therefore $|gA| = |A|$ for all $g \in G$. □

Claim: The cosets form a partition of G. We wish to show $g_1A = g_2A$ or $g_1A \cap g_2A = \emptyset$. Suppose $g_1A \cap g_2A \neq \emptyset$. Then there are $a_1, a_2 \in G$ such that

$$\begin{aligned} g_1 a_1 &= g_2 a_2 \\ g_1 &= g_2 \underbrace{a_2 a_1^{-1}}_{\in A} \in g_2 A \end{aligned}$$

Thus $g_1A \subseteq g_2A$. Similarly $g_1A \supseteq g_2A$, so $g_1A = g_2A$. □

Now if n is the number of cosets,

$$|G| = \sum_{i=1}^{n} |\underbrace{g_i A}_{\text{disjoint}}| = n|A|$$

therefore $|A|$ divides $|G|$. □

9. Prove that any uncountable subset of the reals has a limit point.
   To prove this, first we prove the Bolzano-Weierstraß theorem and a couple of basic set theory results.

   THEOREM 5 (Bolzano-Weierstraß). *A bounded infinite subset of* $\mathbb{R}^n$ *has a limit point.*

   Proof: Let X be our infinite set bounded by M, i.e., for all $x = (x_1, \ldots, x_n) \in X$, $|x_i| < M$. Let $C^0 = \{z \in \mathbb{R}^n : |z_i| \leq M\}$. Note that $X \in C^0$. Let $\{C_i^1\}_{i=1}^{2^n}$ be the cubes of side length M subdividing C^0. There is an $i_1 \in \{1, \ldots, 2^n\}$ such that $C_{i_1}^1$ has infinitely many points of X. Repeat this subdivision process an infinitum. The intersection of a decreasing sequence of closed sets in a complete space (such as $\mathbb{R}^n$) contains at least one point, say x_0. Since each open set around x_0 contains some box $C_{i_m}^m$ [by choosing m large enough] containing in turn infinitely many points of X, we see that x_0 is a limit point of X. □

   THEOREM 6 (Cantor's Diagonal Argument). $\mathbb{Q}$ *is countable.*

   This equivalently shows that $\mathbb{N} \times \mathbb{N}$ is countable.

   Proof: Consider each rational number as specified by its numerator and denominator.

numerator denominator	1	2	3	4	...
1	1	3	4	10	
2	2	5	9		
3	6	8			
4	7				

   Thus

$$\begin{aligned} 1 &\to (1,1) \\ 2 &\to (2,1) \\ 3 &\to (1,2) \\ 4 &\to (1,3) \\ 5 &\to (2,2) \\ 6 &\to \ldots \end{aligned}$$

□

   THEOREM 7. *The countable union of countable sets is countable.*

   Proof: Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of countable sets with $c_i : \mathbb{N} \to A_i$, each onto. Let $d : \mathbb{N} \times \mathbb{N} \to \cup_{i=1}^{\infty} A_i$ by $d(i,j) = c_i(j)$. Let $e : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by Cantor's diagonal argument. Each of d and e are onto, so $d \circ e : \mathbb{N} \to \cup_{i=1}^{\infty} A_i$ is onto, therefore $\cup_{i=1}^{\infty} A_i$ is countable. □

PROOF. Let X be our uncountable subset of $\mathbb{R}$. Consider $\{B_r\}_{r\in\mathbb{Q}}$ where $B_r = \{x \in \mathbb{R} : |x - r| \leq 1\}$. This forms a countable closed cover of $\mathbb{R}$.

> *Claim: There is an $r \in \mathbb{Q}$ such that $B_r \cap X$ is uncountable.* Suppose not. Then there are only countably many $x \in X$ in each B_r. This means $X \subseteq \mathbb{R} \subseteq \cup_{r\in\mathbb{Q}} B_r$ is countable [prior result], which is a contradiction. Therefore such an r exists. □

Now $B_r \cap X$ is an infinite bounded subset of $\mathbb{R}$, so by applying the Bolzano-Weierstraß theorem, we see X has a limit point. □

10. Prove the harmonic series diverges.

PROOF.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{n} \\
&\geq \lim_{k\to\infty} \sum_{n=1}^{f(k)} \frac{1}{n} \text{ where } f(k) = \text{ largest power of 2 less than or equal to } k \\
&= \lim_{k\to\infty} \left[1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{f(k)-1}+1} + \ldots + \frac{1}{2^{f(k)}}\right)\right] \\
&\geq \lim_{k\to\infty} \left[1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{f(k)}} + \ldots + \frac{1}{2^{f(k)}}\right)\right] \\
&= \lim_{k\to\infty} \left[1 + \frac{f(k)}{2}\right] = \infty
\end{aligned}$$

□